Interaction time of Korteweg–de Vries solitons

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The interaction time of Korteweg–de Vries solitons is studied by using Konno and Ito's complex-time-plane method [J. Phys. Soc. Jpn. **56**, 987 (1987)]. We find that the behavior of the interaction time reflects the particle-wave dual nature of the soliton. Most of this feature is explained by the rectangular model of Aossey *et al.* [Phys. Rev. A **45**, 2606 (1992)].

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Solitons are not only a very powerful concept in the various fields where nonlinear problems arise, but are also a useful communicating tool due to their exchange properties, i.e., no profile changes after collisions. The further advantage of the solitons in the communication system is that solitons are spontaneously formed from the initial pulses during their propagation in nonlinear media. These spontaneous formations of the solitons have been discussed in the initial condition problem and have been well established [1,2]. However, the soliton *formation time* or the *interaction time* have not been discussed so far. These times are important when the system size is reduced. In this paper, we study the interaction time for two colliding Korteweg–de Vries (KdV) solitons with various amplitude differences.

We apply polar representations [3] of the soliton to define the soliton interaction time. Konno and Ito [4] have investigated nonlinear interactions between solitons for KdV and Boussinesq equations in terms of the behavior of poles of the soliton solutions in the complex time plane. Before defining the interaction time, we summarize the complex-time-plane method.

The KdV equation we consider is given by

$$u_t + 12uu_x + u_{xxx} = 0. \tag{1}$$

One soliton solution with a wave number k and a positive constant A is derived from an auxiliary function

$$\phi(x,t) = 1 + Ae^{kx-\beta t}. \tag{2}$$

Here, $\beta = k^3$. Simple zeros of ϕ are placed at $t_n(k) = t_{R_n} + it_{I_n}$ $(n = 0, \pm 1, \pm 2, \ldots)$ where

$$t_{R_n} = \frac{kx+\delta}{\beta}, \tag{3}$$

$$t_{I_n} = \frac{(2n+1)\pi}{\beta}, \tag{4}$$

with $\delta = \ln A$. By using the zeros the soliton solution is expressed as

$$u = -\left(\frac{k}{\beta}\right)^2 \sum_{n=-\infty}^{\infty} \frac{1}{[t - t_n(k)]^2}. \tag{5}$$

Real and imaginary parts of complex time describe a trajectory and an amplitude of the soliton, respectively. In two-soliton interactions, the auxiliary function is described by

$$\phi(x,t) = 1 + A_1e^{\eta_1} + A_2e^{\eta_2} + A_3e^{\eta_1+\eta_2}, \tag{6}$$

where

$$\eta_j = k_jx - \beta_jt, \tag{7}$$

$$\beta_j = k_j^3, \tag{8}$$

$$A_3 = \left(\frac{k_1-k_2}{k_1+k_2}\right)^2 A_1A_2, \tag{9}$$

and A_j and k_j are positive constants. Typical two-soliton interactions in the complex time plane are shown in Fig. 1. The zeros deviate from the noninteracting values in both the x-t_R and x-t_I planes during the interaction. The interaction center can be easily found in the inset of Fig. 1(a) where the zeros related to transfer concentrate to a single zero. The beginning and ending of the interaction can be observed in the x-t_I plane [Fig. 1(b)] as the beginning and ending of the deviation from the noninteracting values $(2n+1)\pi/\beta(n$=integer$)$. Therefore, we can get the interaction *length* as the interval of these deviations in the x-t_I plane and then define the *interaction time* via the x-t_R relation by using this length.

Now let us consider the physical picture of this definition. Soliton amplitude changes during the interaction. As we mentioned above, the imaginary part of complex time describes an amplitude of the soliton. We use this property of the soliton in the interacting regions. Of course, one can directly calculate the interaction time by using the changes of the soliton amplitude in real space. The advantage of Konno and Ito's technique is that it gives greater sensitivity than this direct one because the amplitudes are divided into many branches in the x-t_I plane and the changes due to the interaction appear in the branches with $t_I(k_1) \sim t_I(k_2)$. Then we can easily extract the most sensitive branch of the interaction from the branches.

Figure 2 shows dependence of the interaction time, determined by this definition, on α $(= k_1/k_2)$. As α increases, the interaction time first decreases and reaches a minimum at $\alpha = \sqrt{3}$, and then increases. These behaviors reflect the dual nature of the soliton. The decreasing part of the interaction time mainly comes from the particle nature of solitons. In the particle-interaction picture,

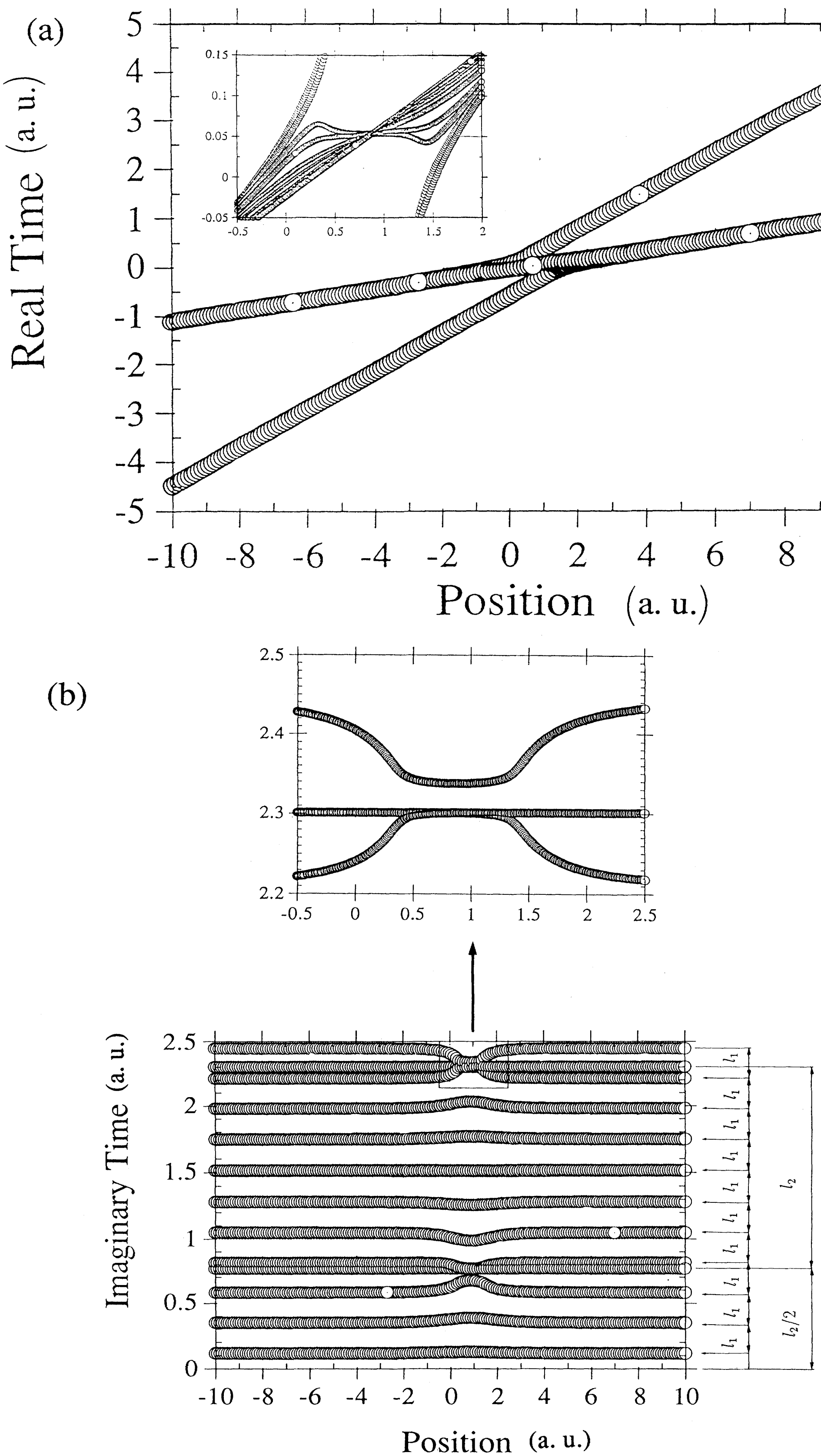

FIG. 1. Typical trajectories of zeros. (a) Real time vs position; (b) imaginary time vs position ($l_1 = \pi/\beta_1, l_2 = \pi/\beta_2$). The parameters are $k_1 = 3.0, k_2 = 1.6, A_1 = 1.0, A_2 = 1.0$.

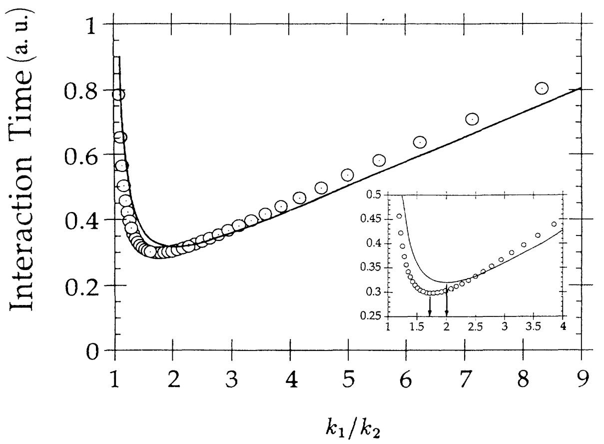

FIG. 2. Interaction time vs k_1/k_2. The circles are the results of our numerical experiment and the solid line is given by Eq. (11). The magnitude of the line is multiplied by 10 in order to compare with our numerical results.

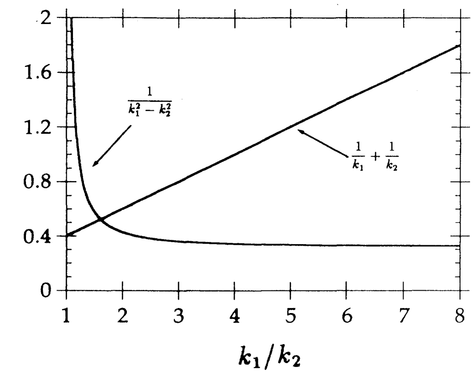

FIG. 4. $v_1 - v_2$ and $W_1 + W_2$ vs k_1/k_2. (The value of k_1 is fixed at 5.0; k_2 varies.)

the interaction time generally decreases as the relative velocity of the interacting particles increases. However, in the particle picture the behavior of the interaction time for $\alpha > \sqrt{3}$, where it increases with α, appears anomalous. This anomaly is related to the character of the soliton as an extended object. The parameter k represents not only an amplitude and a velocity (k^2) of the soliton but also the soliton width. For the KdV soliton, the width is inversely proportional to the square root of the velocity. This means that the spatial interaction regions are extended as the soliton velocity decreases. Therefore, the interaction time between solitons increases with α due to the soliton wave nature. The minimum can be considered as the balancing point of these two natures. It is interesting that the α value giving the minimum interaction time ($\alpha = \sqrt{3}$) is consistent with the single peak formation limit at the interaction center. According to the analytical investigations of the soliton profile at the interaction center given in Ref. [5], the results show that a single peak is formed if $\alpha \geq \sqrt{3}$, while double peaks are formed if $1 < \alpha < \sqrt{3}$. These are shown in Fig. 3.

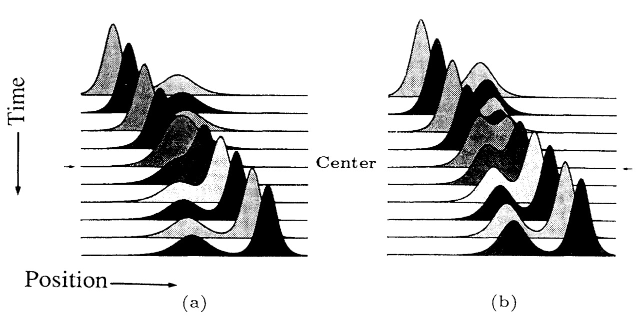

FIG. 3. Time evolution of two solitons from $t = -5$ to 5 for $k_1 = 3.0, A_1 = A_2 = 1.0$, and (a) $k_2 = 1.0$ and (b) $k_2 = 1.6$.

These behaviors can be explained by the simple model of Aossey *et al.* [6]. They consider the solitons to be rectangular pulses with an amplitude k^2 and a width $1/k$. The interaction length (L) satisfies the following relation:

$$L = (v_1 - v_2)\Delta T \geq W_1 + W_2, \tag{10}$$

where v_j and W_j are the velocity and the width of jth soliton, respectively. Since solitons exchange their positions after an overtaking collision, the term $W_1 + W_2$ representing the sum of the two soliton's width describes the minimum interaction region. From Eq. (10), the interaction time ΔT of solitons is given by

$$\Delta T \geq \frac{W_1 + W_2}{v_1 - v_2} = \frac{1}{k_1^2 - k_2^2}\left(\frac{1}{k_1} + \frac{1}{k_2}\right). \tag{11}$$

From this equation, it turns out that the interaction time is comprised of two parts. One is the relative velocity $v_1 - v_2$ representing the particle nature of the soliton and second is the term $W_1 + W_2$ expressing the character of the soliton as an extended object. Figure 4 shows the contribution of each part to ΔT. The decreasing and increasing parts of ΔT in Fig. 2 correspond to the terms $v_1 - v_2$ and $W_1 + W_2$, respectively. In this way, the main feature of Fig. 2 can be understood from Eq. (11). However, the α value of the minimum interaction time in this model is not in agreement with that in the numerical experiment. This disagreement comes from the rectangular pulse approximation.

In summary, we have studied the interaction time of two colliding KdV solitons with various amplitude differences by using Konno and Ito's complex-time-plane method. The behavior of the interaction time reflects the dual nature, i.e., particle and wave, of the soliton. This can be confirmed by the simple model of Aossey *et al.*

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